BGD COLLEGE ,KESAIBAHAL

Blended learning modules

1st Year 1st SEM

Subject and paper :- **MATHEMATICAL PHYSICS-I (PAPER-I)**

Fun^n of more than one Variable:-

Limits:- If $z = f(x,y)$ be a fun^n of two variable then it is said to have limit L as $x \to a$, $y \to b$

$$\boxed{\therefore \lim_{(x,y)\to(a,b)} f(x,y) = L}$$

$\lim_{\substack{x\to a\\ y\to b}} f(x,y) = l$, $\lim_{\substack{x\to a\\ y\to b}} g(x,y) = m$

(1) $\lim_{\substack{x\to a\\ y\to b}} (f \pm g) = L \pm m$ (2) $\lim_{\substack{x\to a\\ y\to b}} (f \cdot g) = Lm$

(3) $\lim_{\substack{x\to a\\ y\to b}} \left(\frac{f}{g}\right) = \frac{L}{m}$ if $m \neq 0$

Continuity:- The fun^n $f(x,y)$ is said to be continuous at point (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

Working of Limit:-

St-1 find $f(x,y)$ along $x \to a$ & $y \to b$

St-2 find $f(x,y)$ along $y \to b$ & $x \to a$

St-3 If $a \to 0$, $b \to 0$ find limit along $y = mx$ or $y = mx^2$.

Ex-1 Find Limit $\lim_{(x,y)\to(0,0)} \frac{xy}{y^2 - x^2}$

Sol^n $\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy}{y^2 - x^2} = \lim_{y\to 0} 0 = 0$

$\lim_{x\to 0} = 0$

$y = mx$ along the line

$\lim_{x\to 0} \frac{mx^2}{m^2x^2 - x^2} = \lim_{x\to 0} \frac{x^2 m}{x^2(m^2-1)} = \frac{m}{m^2-1}$ $\therefore$ limit doesn't exist

$\therefore$ it differ from value of m

Ex-2 find the value of $\lim_{\substack{x\to 1\\ y\to 2}} \frac{2x^2y}{x^2+y^2+1}$

$\lim_{\substack{x\to 1\\ y\to 2}} \frac{2x^2y}{x^2+y^2+1} = \lim_{x\to 1} \frac{2\cdot 2x^2}{x^2+4+1} = \frac{4x^2}{x^2+5} = \frac{4}{6} = \frac{2}{3}$

Ex-4 Show that the fun $f(x,y) = x^2+2y \quad (x,y) \neq (1,2)$
$= 0 \quad , (x,y) = (1,2)$ is continuous at $(1,2)$

$$\lim_{\substack{x\to1\\y\to2}} f(x,y) = \lim_{\substack{x\to1\\y\to2}} (x^2+2y) = \lim_{y\to2}(1+2y) = 5$$

$$\lim_{\substack{x\to1\\y\to2}} (x^2+2y) = \lim_{x\to1}(x^2+4) = 5$$ But $f(x,y)=0$ $x=1, y=2$.

PARTIAL DERIVATIVES :-

If $z = f(x,y)$ then derivative of z w.r.t x keeping const is called partial derivative of z w.r.t x

$$\frac{dz}{dx} = \lim_{\delta x\to0} \frac{f(x+\delta x, y) - f(x,y)}{\delta x} = f_x(x,y)$$

Also derivative of z w.r.t y keepin x const.

$$\frac{dz}{dy} = \frac{f(x, y+\delta y) - f(x,y)}{\delta y} = f_y(x,y).$$

PARTIAL DERIVATIVES OF HIGHER ORDER :-

If $z = f(x,y)$ then $\frac{d}{dx}\left(\frac{dz}{dy}\right) = \frac{d^2z}{dxdy}$ or f_{xy}

$\frac{d}{dy}\left(\frac{dz}{dy}\right) = \frac{d^2z}{dy^2}$ or f_{yy}

Ex1. find first order derivatives of $u = e^x\sin y$

sol^n :- $u = e^x \sin y$

$\Rightarrow u_x = \frac{d(e^x\sin y)}{dx}$ $\qquad u_y = \frac{d\, e^x\sin y}{dx}$

$= e^x \sin y$ $\qquad = e^x\cos y.$

Ex2

if $z = \ln(x+\sqrt{x^2-y^2})$ find first order derivatives.

$z = \ln(x+\sqrt{x^2-y^2})$

$$\therefore \frac{dz}{dx} = \frac{1}{x+\sqrt{x^2-y^2}}\left(1+\frac{1}{\cancel{2}\sqrt{x^2-y^2}}(\cancel{2}x)\right)$$

$$= \frac{1}{x+\sqrt{x^2-y^2}}\left(1+\frac{x}{\sqrt{x^2-y^2}}\right)$$

$$\frac{dz}{dy} = \frac{1}{x+\sqrt{x^2-y^2}}\left(\frac{1}{\cancel{2}\sqrt{x^2-y^2}}(-\cancel{2}y)\right)$$

$$= \frac{-y/\sqrt{x^2-y^2}}{x+\sqrt{x^2-y^2}}$$

variable to be Treated as constant :-

If $z = x^2 + 2y^2$ and $x = r\cos\theta$, $y = r\sin\theta$. Find $\left(\frac{dz}{dy}\right)_r$, $\left(\frac{dz}{d\theta}\right)_x$

we have $x^2 + y^2 = r^2$

$\tan\phi = \frac{y}{x} \Rightarrow y = x\tan\theta$

$\therefore z = x^2 + 2y^2$

putting $x^2 = r^2 - y^2$

we get $z = r^2 - y^2 + 2y^2$

$\Rightarrow z = r^2 + y^2 \Rightarrow \left(\frac{dz}{dy}\right) = 2y$

Also $z = x^2 + 2y^2$

$= x^2 + 2x^2 \tan^2\theta$

$\left(\frac{dz}{d\theta}\right) = 2x^2 (2\tan\theta)\sec^2\theta = 4x^2\tan\theta \frac{r^2}{x^2} \left(\sec\theta = \frac{r}{x}\right)$

$= 4r^2 \tan\theta$

Total Derivative:-

if $z = f(x, y)$ be funⁿ of 2 variables then total diff

is represent as $dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$

Similarly $f = f(x, y, z)$ then $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$

$$\frac{du}{dt} = \frac{du}{dx} \cdot \frac{dx}{dt} + \frac{du}{dy} \cdot \frac{dy}{dt}.$$

Q1 if $u = u\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$ PT $x^2 \frac{du}{dx} + y^2 \frac{du}{dy} + z^2 \frac{du}{dz} = 0.$

let $s = \frac{y-x}{xy} = \frac{1}{x} - \frac{1}{y}$ so $\frac{ds}{dx} = \frac{1}{x^2}$, $\frac{ds}{dy} = \frac{1}{y^2}$, $\frac{ds}{dz} = 0$

$t = \frac{z-x}{xz} = \frac{1}{x} - \frac{1}{z}$ $\frac{dt}{dx} = -\frac{1}{x^2}$, $\frac{dt}{dy} = 0$, $\frac{dt}{dz} = \frac{1}{z^2}$

we have

$u = u(s, t)$

$\Rightarrow \frac{du}{dx} = \frac{du}{ds} \cdot \frac{ds}{dx} + \frac{du}{dt} \cdot \frac{dt}{dx} = \frac{du}{ds}\left(-\frac{1}{x^2}\right) + \frac{du}{dt}\left(-\frac{1}{x^2}\right)$

$\Rightarrow x^2 \frac{du}{dx} = -\frac{du}{ds} - \frac{du}{dt}$ ——— (1)

Similarly $\frac{du}{dy} = \frac{du}{ds} \cdot \frac{ds}{dy} + \frac{du}{dt} \cdot \frac{dt}{dy} = \frac{du}{ds}\left(\frac{1}{y^2}\right) + \frac{du}{dt} \cdot 0$

$y^2 \frac{du}{dy} = \frac{du}{ds}$ ——— (2).

Now $\frac{du}{dz} = \frac{du}{ds}\cdot\frac{ds}{dz} + \frac{du}{dt}\cdot\frac{dt}{dz} = \frac{du}{ds}(0) + \frac{du}{dt}\left(\frac{1}{z^2}\right)$

$= z^2\frac{du}{dz} = \frac{du}{dt}$ — (3)

adding eq^n 1, 2, & 3

$$x^2\frac{du}{dx} + y^2\frac{du}{dy} + z^2\frac{du}{dz} = -\frac{du}{ds} - \frac{du}{dt} + \frac{du}{ds} + \frac{du}{dt}.$$

$$x^2\frac{du}{dx} + y^2\frac{du}{dy} + z^2\frac{du}{dz} = 0.$$

Euler's Theorem:-

If u is a homogeneous fun^n of degree n in x and y then Euler's theorem is $x\frac{du}{dx} + y\frac{du}{dy} = nu$.

Ex-1

$u = \sin^{-1}\frac{x^2+y^2}{x+y}$ PT $x\frac{du}{dx} + y\frac{du}{dy} = \tan u$.

let $f = \sin u = \frac{x^2+y^2}{x+y} = \frac{x^2(1+y^2/x^2)}{x(1+y/x)} = x\phi(y/x)$ $n=1$

from Euler's theorem $x\frac{df}{dx} + y\frac{df}{dy} = nf$.

$$\Rightarrow x\cos u\frac{du}{dx} + y\cos u\frac{du}{dy} = f$$

$$\Rightarrow \cos u\left[x\frac{du}{dx} + y\frac{du}{dy}\right] = \sin u.$$

$$\Rightarrow x\frac{du}{dx} + y\frac{du}{dy} = \frac{\sin u}{\cos u} = \tan u.$$

Ex 2

$u = \log(x^2+xy+y^2)$ PT $x\frac{du}{dx} + y\frac{du}{dy} = 2$.

sol^n let $f = e^u = x^2+xy+y^2 = x^2\left(1+\frac{xy}{x^2}+\frac{y^2}{x^2}\right)$

$= x^2\left(1+\frac{y}{x}+\frac{y^2}{x^2}\right) = x^2\phi\left(\frac{y}{x}\right)$

degree = 2

Applying Euler's th^m : $x\frac{df}{dx} + y\frac{df}{dy} = nf$

$$\Rightarrow xe^u\frac{du}{dx} + ye^u\frac{du}{dy} = 2e^u \quad \{f = e^u\}$$

$$\Rightarrow x\frac{du}{dx} + y\frac{du}{dy} = 2.$$

Differential ...

Exact Diff. Eqn:-

if the eqn is in form of $M(x,y)\,dx + N(x,y)\,dy = 0$ is said to be exact diff eqn. condition $\boxed{\frac{dM}{dy} = \frac{dN}{dx}}$

Ex 1 $y\,dx + x\,dy = 0$

$M dx + N dy = 0$

$M = y \quad N = x$

$\frac{dM}{dy} = 1 \quad \frac{dN}{dx} = 1$

$\therefore \frac{dM}{dy} = \frac{dN}{dx} \Rightarrow$ The eqn is exact.

Integrating factor:-

1) $x\,dy + y\,dx = d(xy)$

2) $\frac{y\,dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right)$

3) $\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right)$

4) $\frac{y\,dx - x\,dy}{x^2+y^2} = d\left(\tan^{-1}\frac{x}{y}\right)$

5) $\frac{y\,dx - x\,dy}{xy} = d\left(\log\frac{x}{y}\right)$

(6) : $\frac{y\,dx - x\,dy}{x^2y^2} = d\left(\frac{1}{xy}\right)$

(7) $\frac{ye^x\,dx - e^x\,dy}{y^2} = d\left(\frac{e^x}{y}\right)$

(8) $\frac{2x\,dx + 2y\,dy}{x^2+y^2} = d\left[\log(x^2+y^2)\right]$

(9) $\frac{2xy\,dx - x^2\,dy}{y^2} = d\left(\frac{x^2}{y}\right)$

(10) $\frac{2xy^2\,dx - 2yx^2\,dy}{y^4} = d\left(\frac{x^2}{y^2}\right)$.

Integrating factor of Non Exact Diff EqN:-

Rule 1 if $Mx + Ny \neq 0$ then $\frac{1}{Mx+Ny}$ is IF.
$M dx + N dy = 0$

Rule 2 if $M = y f_1(xy)$, $N = x f_2(xy)$

$DF = f_1(xy)\,y\,dx + f_2(xy)\,x\,dx = 0$

then $\frac{1}{Mx - Ny}$ is IF.

Rule 3 if $\left(\frac{dM}{dy} - \frac{dN}{dx}\right)/N$ in a fun^n of x only $= F(x)$ then IF

Rule 4 $\left(\frac{dN}{dx} - \frac{dM}{dy}\right)/M$ is fun^n of y $\therefore F(y)$ is IF;

$IF = e^{\int F(y)\,dy}$.

Rule-V :- if. DE is in the form of $x^a y^b (my\,dx + nx\,dy) + x^{a'} y^{b'} (m'y\,dx + n'x\,dy) = 0$ the IF $= x^h y^k$.

LAGRANGE'S METHOD :-

1- if $F(x,y,z) = f(x,y,z) + \lambda \phi(x,y,z)$

2- Parti diff $F(x,y,z)$ wrt x,y,z equat them to zer

$\frac{dF}{dx} = 0$, $\frac{dF}{dy} = 0$ $\frac{dF}{dz} = 0$.

3- Solve four eq^n $\frac{dF}{dx} = 0$ $\frac{dF}{dy} = 0$ $\frac{dF}{dz}$ and $\phi(x,y,z) = 0$.

to obtained value of x y & z.

Ex-1

find minimum value of $x^2+y^2+z^2$ condition $xyz = a^3$

$f(x,y,z) = x^2+y^2+z^2$.

contraint eq^n is $\phi(x,y,z) = xyz - a^3$

$\therefore$ A.E $F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$

$F(x,y,z) = x^2+y^2+z^2 + \lambda(xyz - a^3)$ —— (1)

Then $\frac{dF}{dx} = 0$ gives $2x + \lambda yz = 0$ — (2)

$\frac{dF}{dy} = 0$ gives $2y + \lambda xz = 0$ —— (3)

$\frac{dF}{dz} = 0$ gives $2z + \lambda xy = 0$ —— 4.

from eq^n (2) (3) & (4) we get $2x = -\lambda yz$ or $2x^2 = -\lambda xy$

$2y = -\lambda xz$, $2z = -\lambda xy$

$\therefore$ putting the value of x, y, z in $xyz = a^3$

$x = a$ $y = a$ $z = a$.

Hence min^m value of funct is $x^2+y^2+z^2 = 3a^2$.

Vector Algebra

Vector < both magnitude, dirⁿ

Scalar — Mag

$\vec{A}$ $|\vec{A}|$. magnitude

- **equal Vector:**

$\vec{A}$ & $\vec{B}$

same dirⁿ & magn. $\longrightarrow \vec{A}$

$|\vec{A}| = |\vec{B}|$ $\longrightarrow \vec{B}$.

- **Negative vector**

equal magnitude but oposite dirⁿ

$|\vec{A}| = |-\vec{B}|$ $\longrightarrow \vec{A}$

$\vec{A} = -\vec{B}$. $\longleftarrow \vec{B}$

- **Null vector**

if magnitude is zero and dirⁿ – arbitrary

- **Co-linear Vector**

Parallel vector – same

&

Anti par – dirⁿ oposi

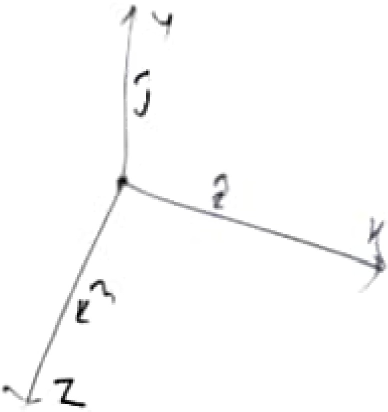
magnitude may be equal or not

Unit vector

$\vec{A}$

$\hat{A} = \frac{\vec{A}}{|\vec{A}|}$ A vect m.

$\hat{A} = \frac{\vec{A}}{A}$



Like Vectors

Polar Vector

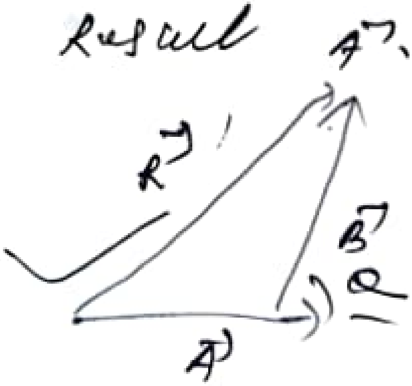
Addition of two Vectors.

Let $\vec{A}, \vec{B}$.

Result

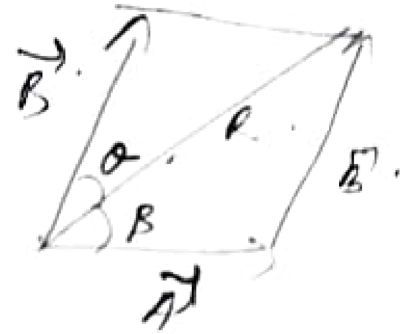
$\vec{R} = \vec{A} + \vec{B}$

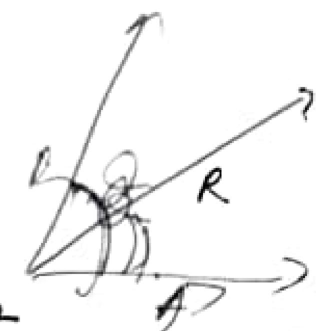
Triangle law of Vector Addn.



Law of Vector Addition

90%





$R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$

$\tan\beta = \dfrac{B\sin\theta}{A + B\cos\theta}$

$\beta = \tan^{-1}\left(\dfrac{B\sin\theta}{A + B\cos\theta}\right)$

$\dfrac{\vec{A}\ \&\ \vec{B}}{\theta}$

$\beta \ (\vec{A}\ \&\ \vec{R})$

$\dfrac{1}{\tan} = \tan^{-1}$

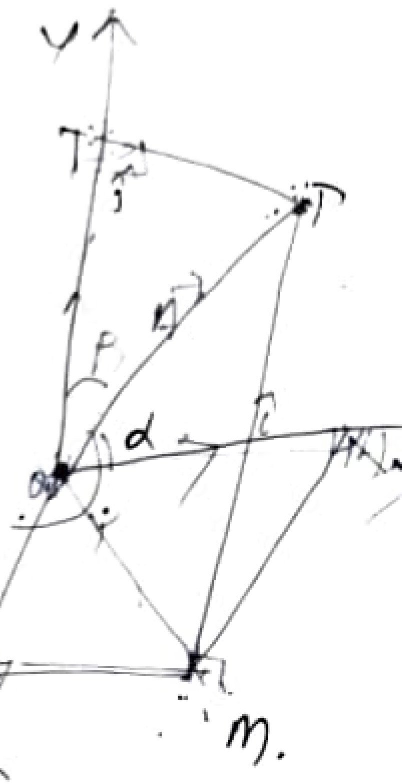
$\dfrac{1}{a^n} = a^{-n}$

$\vec{A} + \vec{B} = \vec{B} + \vec{A}$

$\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$

$= \vec{A} + \vec{B} + \vec{C}$

commutative law
&
associative



$\vec{OP} = \vec{r}$

$\overline{ON} = \vec{A_x}$

$OT = \vec{A_y}$

$OS = \vec{A_z}$

$\vec{OP} = \vec{OM} + \vec{OT}$

or $OM + \overline{OT} = \vec{OP}$ —(1)

$OS + SM = OM$

$OS + ON = OM$ —(2)

$\vec{OP} = \vec{OS} + \overline{ON} + \overline{OT}$

$= \vec{A} = \vec{A_x} + \vec{A_y} + \vec{A_z}$

$\vec{A_x} = 0$

$r = \sqrt{x^2 + y^2 + z^2}$

$\frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|}$

$(3, 3, -2)$

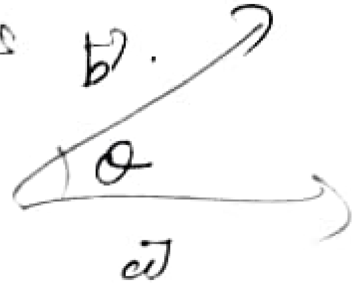
$\vec{A} \cdot \vec{B} = AB \cos\alpha$ Scalar product

$i \cdot j = 0$
$j \cdot k = 0$
$k \cdot i = 0$

$i \cdot i = 1$
$j \cdot j = 1$
$k \cdot k = 1$

$\vec{a}$ & $\vec{b}$ be are two non-zero vectors
& θ be the ap

Scalar product of two vectors



$\vec{a}.\vec{b}$ is dn.

$|\vec{a}|\,|\vec{b}|\cos\theta$.

$\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|\cos\alpha$ dot pr.

$\vec{a} = \hat{i} + 2\hat{j} - 2\hat{k}$

$\vec{b} = 8\hat{i} + \hat{j}$.

Scalar compo b on $\vec{a}$

$\cos\theta . \hat{a}$ $\quad 10/2$

$= |\vec{b}| . \frac{\vec{a}.\vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{\vec{a}.\vec{b}}{|\vec{a}|}$.

$\frac{\vec{a}.\vec{b}}{|\vec{a}|\,|\vec{a}|}$

$\frac{8+2-0}{\sqrt{9}} = \frac{10}{3}$

Vector $\vec{b}$ on $\hat{a}$

$|\vec{b}|\cos\theta . \hat{a}$

$\sqrt{65}$

$\vec{a}$ on $\vec{b}$ where

$\vec{a} = \hat{i} - \hat{j} - \vec{k}$

$\vec{b} = 3\hat{i} + \hat{j} + 3\vec{k}$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{3-1-3}{\sqrt{3}\sqrt{19}} = -\frac{1}{\sqrt{3}\sqrt{19}}$$

scalar projection $\vec{a}$ on $\vec{b}$

$$= |\vec{a}|\cos\theta = \sqrt{3}\,\frac{1}{\sqrt{3}\sqrt{19}} = \frac{-1}{\sqrt{19}}$$

vect pn $\vec{a}$ on $\vec{b}$

$$= \underline{|\vec{a}|\cos\theta \cdot \hat{b}}$$

$$= -\frac{1}{\sqrt{19}} \cdot \frac{\vec{b}}{|\vec{b}|} = \frac{1}{\sqrt{19}} \cdot \frac{3\hat{i}+\hat{j}+3\vec{k}}{\sqrt{19}}$$

$$\vec{a}\cdot\vec{b} = 0 \qquad \frac{-3\hat{i}-\hat{j}-3\vec{k}}{19}$$

$\vec{AB} = -2\hat{i} - 3\hat{j} - 4\hat{k}$

$\vec{CD} = 3\hat{i} + 6\hat{j} - 6\hat{k}$

$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a}$

$|\vec{a}| = |\vec{b}| = |\vec{c}|$

$\vec{AB}\cdot\vec{CD} = 6$

$-6 - 18 + 24 = 0$

are

$\vec{AB} \perp$ to $\vec{CD}$

12 / 20

accu... Vector ...

$\vec{r}$ = Anusfration alpha sendrome.

$\vec{a}+\vec{b}+\vec{c}=0$

$\vec{a} = -(\vec{b}+\vec{c})$

$\vec{a}\cdot\vec{a} = -(\vec{b}+\vec{c})\cdot\vec{a}$

$|\vec{a}|^2 = -\vec{b}\cdot\vec{a} - \vec{c}\cdot a.$

$|\vec{OA}| = |\vec{OP}| = |\vec{OB}|$ W.F.S

$|\vec{a}| = |\vec{r}| = |-\vec{a}|$

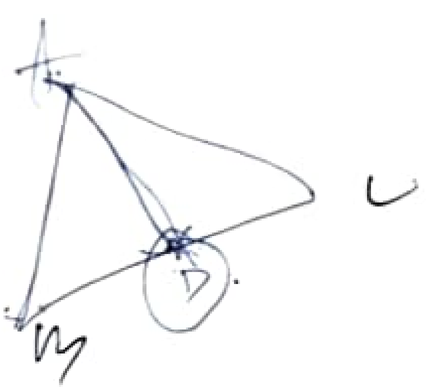
$\vec{AP} = \vec{OP} - \vec{OA} = \vec{r} - \vec{a}.$

$\vec{BP} = \vec{r} + \vec{a}.$

$\vec{AP}\cdot\vec{BP} = (\vec{r}-\vec{a})\cdot(\vec{r}+\vec{a})$

$= \vec{r}\cdot\vec{r} + \vec{r}\cdot\vec{a} - \vec{a}\cdot\vec{r} + \vec{a}\cdot\vec{a}$

$|\vec{r}|^2 - |\vec{a}|^2 = 0$



3 . 5 . 2002

3 . 4 . 2003

...tor products, (cross prod)

$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$

$i\times i = 0 \quad j\times j = 0$

$i\times j = k \quad j\times k = i$

$$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$$

$$\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$$

$$\hat{n} = \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$$

$$\sin\theta = \left|\frac{\vec{a}\times\vec{b}}{\vec{a}\times\vec{b}}\right|$$

$$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}|\times|\vec{b}|} \qquad |\vec{c}| = \sqrt{3}$$

$\vec{a} = i + j + k$

$\vec{b} = 4i + 3j + 4k$

$\vec{c} = i + \alpha j + \beta k$

$$\begin{vmatrix} 1 & 1 & 1 \\ 4 & 3 & 4 \\ 1 & \alpha & \beta \end{vmatrix}$$

$\vec{AB} = \vec{b} - \vec{a}$

$\vec{BC} = \vec{c} - \vec{b}$

$\vec{CA} = \vec{a} - \vec{c}$

$((\vec{a}\times\vec{b}) + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}) \times \vec{AB} = 0$

$(\vec{a}-\vec{d}) \times (\vec{b}-\vec{c})$

$= \vec{a}\times\vec{b} - \vec{a}\times\vec{c} - \vec{d}\times\vec{b} + \vec{d}\times\vec{c}$

$= 0$

$\vec{a}\times\vec{b} = \vec{c}\times\vec{d}$

$\vec{a} = 3i + 2j - \hat{k}$

$\vec{b} = -2i + j + k$

$$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ 3 & 2 & -1 \\ -2 & 1 & 1 \end{vmatrix}$$

$3 - -4$

$= 3i - j + 7\hat{k}$

$|\vec{a}\times\vec{b}| = \sqrt{59}$

$(\vec{a}-\vec{b})\,[\vec{b}-\vec{c} \times (\vec{c}-\vec{a})]$

$(\vec{a}-\vec{b}) \cdot [\vec{b}\times\vec{c} - \vec{b}\times\vec{a} - \vec{c}\times\vec{c} + \vec{c}\times\vec{a}]$

$(\vec{a}-\vec{b})[\vec{b}\times\vec{c} - \vec{b}\times\vec{a} + \vec{c}\times\vec{a}]$

$\vec{a}\cdot(\vec{b}\times\vec{c}) - \vec{a}\cdot(\vec{b}\times\vec{a}) + \vec{a}\cdot(\vec{c}\times\vec{a})$

$-\vec{b}\cdot(\vec{b}\times\vec{c}) + \vec{b}\cdot(\vec{b}\times\vec{a})$

$-\vec{b}\cdot(\vec{c}\times\vec{a})]$

$\vec{a}\cdot(\vec{b}\times\vec{c}) - \vec{b}\,(\vec{c}\times\vec{a}) = 0$

$\vec{AB} = \vec{b} - \vec{a}$

$\vec{BC} = \vec{c} - \vec{b}$

$\vec{CA} = \vec{a} - \vec{c}$

$\left((\vec{a}\times\vec{b}) + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} \right) \times \vec{AB} = 0$

$(\vec{a}-\vec{d}) \times (\vec{b}-\vec{c})$

$= \vec{a}\times\vec{b} - \vec{a}\times\vec{c} \; \cancel{\times} \; - \vec{d}\times\vec{b} + \vec{d}\times\vec{c}$

$= 0$ $\qquad \vec{a}\times\vec{b} = \vec{c}\times\vec{d}$

$\vec{a} = 3i + 2j - k$

$\vec{b} = -2i + j + k$

$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ 3 & 2 & -1 \\ -2 & 1 & 1 \end{vmatrix}$

$3 - -4$

$= 3i - j + 7k$

$|\vec{a}\times\vec{b}| = \sqrt{59}$

$(\vec{a}-\vec{b}) \left[\vec{b}-\vec{c} \times (\vec{c}-\vec{a}) \right]$

$(\vec{a}-\vec{b}) \cdot \left[\vec{b}\times\vec{c} - \vec{b}\times\vec{a} - \vec{c}\times\vec{c} + \vec{c}\times\vec{a} \right]$

$(\vec{a}-\vec{b}) \left[\vec{b}\times\vec{c} - \vec{b}\times\vec{a} + \vec{c}\times\vec{a} \right]$

$\vec{a}\cdot(\vec{b}\times\vec{c}) - \vec{a}\cdot(\vec{b}\times\vec{a}) + \vec{a}\cdot(\vec{c}\times\vec{a})$

$- \vec{b}\cdot(\vec{b}\times\vec{c}) + \vec{b}\cdot(\vec{b}\times\vec{a})$

$- \vec{b}\cdot(\vec{c}\times\vec{a})]$

$\vec{a}\cdot(\vec{b}\times\vec{c}) - \vec{b}\,(\vec{c}\times\vec{a}) = 0$

~~Scalar triple~~ → (4π)

$\vec{a}$, $\vec{b}$ & $\vec{c}$

$\vec{a} \cdot (\vec{b} \times \vec{c})$ or $(\vec{a} \times \vec{b}) \cdot \vec{c}$

$$\boxed{\begin{aligned} \vec{a} &= a_1 \hat{i} + a_2 \hat{j} + a_3 k \\ \vec{b} &= b_1 i + b_2 j + b_3 k \\ \vec{c} &= c_1 i + c_2 j + c_3 k \end{aligned}}$$

$\vec{a} \cdot (\vec{b} \times \vec{c})$
$(\vec{a} \times \vec{b}) \cdot \vec{c}$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$